Measure Concentration and Almost Spherical Sections

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Introduction

§1 Measure Concentration on the Sphere

Main idea of this Section:

We will consider the probability measure on the surface of the unit Euclidean sphere. Informally, for the surface of the sphere S^{n-1} we will have $\mathbb{P}[S^{n-1}] = 1$ and for any $A \subset S^{n-1}$, $P[A]$ will be the probability that a random point of S^{n-1} falls into A.

Measure concentration on the sphere can be approached in two steps.

1. The first step is that for large n, most of S^{n-1} lies quite close to the "equator." (i.e. a hyper-plane going through the center)
2. The second step is to show that the measure on S^{n-1} is concentrated not only around the equator, but near the boundary of any (measurable) subset $A \subset S^{n-1}$ covering half of the sphere.

We will now give a result for the first part which will also be useful in the upcoming sections.

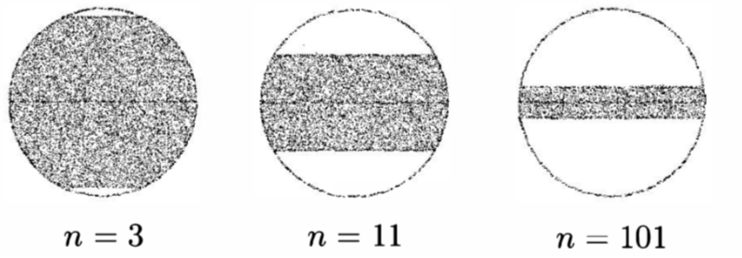
§1 The First Step

Theorem 1.1

For the surface of the sphere S^{n-1} and for a $w > 0$, if we have

$$\mathbb{P}[\{x \in S^{n-1} : -w < x_n < w\}] = 0.9.$$

Then w is of the order $n^{-1/2}$ for large n.



§1 The First Step

Not a proof but a relevant arguement.

We will begin with the observation that for $Z_1, \ldots, Z_n$ $i.i.d.$ standart normally distributed random variables and $Z = (Z_1, \ldots, Z_n)$, then $U = \frac{Z}{\|Z\|}$ is distributed uniformly on the surface of the sphere S^{n-1}. We will now show that for uniformly disributed U, $\|U\|$ is tightly concentrated around $\sqrt{n/3}$

$$\mathbb{E}\|U^2\| = \mathbb{E}[U_1^2 + \cdots + U_n^2] = \sum_{i=1}^{n} \mathbb{E}X_i^2 = \sum_{i=1}^{n} \int_{-1}^{1} \frac{1}{2}x^2 dx = \frac{n}{3}$$

If we assume U_i to be independent, we may show that $\mathbb{P}\left(\left|\|X\|^2 - \frac{n}{3}\right| \geq \epsilon n\right)$, which gives us the result that the volume of a n-dimensional cube tends to lie within $\sqrt{n/3}$ of its centre. Pitman and Ross 2012

Proof.

For a different approach, let's limit the volume of the region where $x_1^2 + x_2^2 < R^2$. From polar coordinates we can get the equation:

$$V_n = \int_0^1 \int_0^{2\pi} V_{n-2}(\sqrt{1-r^2})dr = 2\pi V_{n-2} \int_0^1 (1-r^2)^{(n/2-1)} rdr$$
$$= 2\pi \Big[-\frac{1}{n}(1-r^2)^{n/2} \Big]_0^1 = \frac{2\pi}{n} V_{n-2}$$

Then we want to find the R where:

$$\Big[(1-r^2)^{n/2}\Big]_0^R = 1-(1-R^2)^{n/2} = \frac{1}{2}$$

Therefore we have $R = \sqrt{1-2^{\frac{-4}{n}}} \sim \frac{\sqrt{2\log 2}}{\sqrt{n}}$. □

§1 Second step

Now, we move on to the second step which will be the main result of this section.

Theorem 1.2 (Measure concentration for the sphere)

Let $A \subseteq S^{n-1}$ be a measurable set with with $\mathbb{P}[A] \geq \frac{1}{2}$, and let A_t denote the t- neighbourhood of A, that is, the set of all $x \in S^{n-1}$ whose euclidean distance to A is at most t. then

$$1 - \mathbb{P}[A_t] \leq 2e^{-t^2 n/2}.$$

Few notes:

- there is nothing special about $\frac{1}{2}$, we may have analogous results for $0 < \mathbb{P}[A] < \frac{1}{2}$
- we will prove a slightly weaker version of the theorem with the bound $e^{-t^2 n/4}$.

Theorem 1.3 (Brunn-Minkowski inequality)

let $n \geq 1$ and μ lebesgue measure on $\mathbb{R}^n$. the following inequality holds for $A, B \subset \mathbb{R}^n$

$$[\mu(A+B)]^{1/n} \geq [\mu(A)]^{1/n} + [\mu(B)]^{1/n}$$

we will use a slightly different version of the brunn-minkowski inequlity:

Corollary 1.4

for $A, B \subset \mathbb{R}^n$, the following follows from the Brunn-Minkowski inequlity:

$$\mu\left(\frac{1}{2}(A+B)\right) = \sqrt{\mu(A)\mu(A)}$$

Proof of the Corollary.

The proof follows from the Brunn-Minkowski inequality directly:
$\mu\left(\frac{1}{2}(A+B)\right)^{1/n} \geq \mu(\frac{1}{2}A)^{1/n} + \mu(\frac{1}{2}B)^{1/n}$
$= \frac{1}{2}\left(\mu(A)^{1/n} + \mu(B)^{1/n}\right) \geq (\mu(A)\mu(B))^{1/2n}$ by the inequality
$\frac{1}{2}(a+b) \geq \sqrt{ab}$. □

§1 Proof of the Main Theorem

Proof of Measure Concentration on the Sphere.

For a set $A \subset S^{n-1}$ we define $\tilde{A}$ as the union of all the segments connecting the points of A to 0:

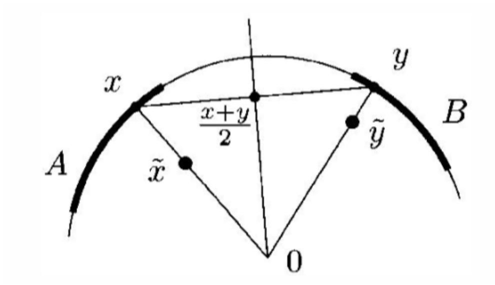
$$\tilde{A} = \{\alpha x : x \in A, \alpha \in [0,1]\} \subseteq B^n$$

Then we have $\mathbb{P}[A] = \mu(\tilde{A})$, where $\mu(\tilde{A}) = vol(\tilde{A})/vol(B^n)$ is the normalized volume of A; in fact, this can be taken as the definition of $\mathbb{P}[A]$. Let $t \in [0,1]$, let $\mathbb{P}[A] \geq \frac{1}{2}$, and let $B = S^{n-1}/A_t$. Then $\|a - b\| \geq t$ for $a \in A$ and $b \in B$.

§1 Proof of the Main Theorem: Lemma

Lemma 1.5

For any $\tilde{x} \in \tilde{A}$ and $\tilde{y} \in \tilde{B}$, we have $\|\frac{\tilde{x}+\tilde{y}}{2}\| < 1 - t^2/8$.



Proof of the Lemma.

Let $\tilde{x} = \alpha x$, $\tilde{y} = \beta y$, $x \in A$, $y \in B$: First we calculate, by the Pythagorean theorem and by elementary calculus,

$$\|\frac{x+y}{2}\| \leq \sqrt{1-\frac{t^2}{4}} \leq 1-\frac{t^2}{8}.$$

For passing to $\tilde{x}$ and $\tilde{y}$ we assume $\beta = 1$. Then

$$\begin{aligned}\|\frac{\tilde{x}+\tilde{y}}{2}\| = \|\frac{\alpha x+y}{2}\| &\leq \alpha\|\frac{x+y}{2}\| + (1-\alpha)\|\frac{y}{2}\| \\ &= \alpha(1-t^2/8) + (1-\alpha)(1-\frac{1}{2}) < 1-\frac{t^2}{8}.\end{aligned}$$

□

§1 Proof of the Main Theorem

Proof(Cont.)

By the lemma, the set $\frac{1}{2}(\tilde{A}+\tilde{B})$ is contained in the ball of radius $1-t^2/8$ around the origin. Applying Brunn-Minkowski to $\tilde{A}$ and $\tilde{B}$, we have

$$\left(1-\frac{t^2}{8}\right)^n \geq \mu(\frac{1}{2}(\tilde{A}+\tilde{B})) \geq \sqrt{\mu(\tilde{A})\mu(\tilde{B})} = \sqrt{P[A]P[B]} \geq \sqrt{\frac{1}{2}P[B]}$$

So,

$$P[B] \leq 2\left(1-\frac{t^2}{8}\right)^{2n} \leq 2e^{-t^2 n/4}.$$

Matousek 2002 □

§2 Isoperimetric Inequalities and More on Concentration

§2 Isoperimetric Inequalities and More on Concentration

Main idea of this Section:

In this chapter we will investigate some isoperimetric inequalities. We will prove the mother of all isoperimetric inequalities and we also prove some cool inequality which has some combinatorical applications.

Theorem 2.1

Among all planar geometric figures with a given perimeter, the circular disk has the largest possible area.

§2 Generalization

Corollary 2.2

We can generalize this inequality into higher dimensions. Also we can generalize this too and it gets easier to prove. We will prove that among all sets of a given volume in some metric space under consideration, a ball of that volume has the smallest volume of the t-neighborhood. Letting $t \to 0$*, one can get a statement involving the perimeter or surface area.*

§2 Euclidean Space Version

We will prove that for any compact set $A \subseteq \mathbb{R}^n$ and any $t \geq 0$, we have $V(A_t) > V(B_t)$, where B is a ball of the same volume as A.

§2 Proof

Proof.

By rescaling, we may assume that B is a ball of unit radius. Then $A_t = A + tB$, and so:

$$V(A_t) = V(A + tB) \geq (V(A)^{1/n} + tV(B)^{1/n})^n = (1+t)^n V(B) = V(B_t)$$

□

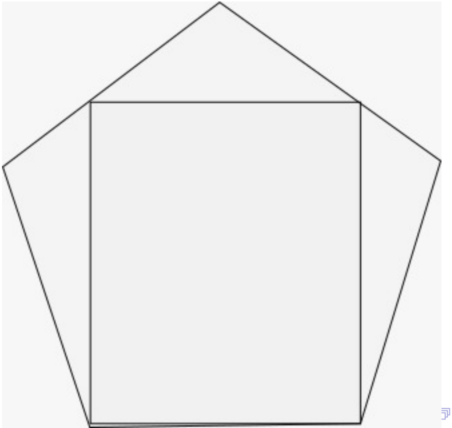
§2 Different Inequality

For the sphere S^{n-1} with the usual Euclidean metric inherited from $\mathbb{R}^n$, an r-ball is a spherical cap, i.e. , an intersection of S^{n-1} with a half-space. The isoperimetric inequality states that for all measurable sets $A \subseteq S^{n-1}$ and all $t > 0$, we have $P[A_t] > P[C_t]$, where C is a spherical cap with $P[C] = P[A]$. We are not going to prove this; no really simple proof seems to be known.

§2 Some Interesting Inequality

Theorem 2.3

Let $\mathbb{A}$ be a convex $n-$gon and $\mathbb{B}$ be a convex $m-$gon. Also assume that $\mathbb{B} \subseteq \mathbb{A}$ i.e. $\mathbb{B}$ completely lies inside the $\mathbb{A}$. Then the perimeter of $\mathbb{A}$ is greater or equal than perimeter of $\mathbb{B}$.



§2 Proof

Proof.

Let's denote area with $A(\cdot)$ and perimeter with $p(\cdot)$. Since $\mathbb{B} \subseteq \mathbb{A}$ we also have $\mathbb{B}_t \subseteq \mathbb{A}_t$. Therefore $A(\mathbb{B}_t) \leq A(\mathbb{A}_t)$. Now let's calculate $A(\mathbb{A}_t)$. We can see the shape of $\mathbb{A}_t$ consists of $\mathbb{A}$, n rectangles with the lengths equal to sides of $\mathbb{A}$ and height equal to t and we have n cones of circles at the vertices with the radius t and the angle with outer angles of $\mathbb{A}$. Area of rectangles equal to $tp(\mathbb{A})$. Area of cones of circles are equal to $t^2\pi$ because for any polygon sum of the outer angles is equal to 2π. Then for all $t \geq 0$ we have:

$$A(\mathbb{B}) + t\,p(\mathbb{B}) + t^2\pi \leq A(\mathbb{A}) + t\,p(\mathbb{A}) + t^2\pi.$$

Since this equality holds for all $t > 0$ if we choose t large enough we can see that $p(\mathbb{B}) \leq p(\mathbb{A})$ must hold. □

§2 Combinatorial Application

Proposition 2.1

There are $(n+1)^2$ points lying inside a square with side length n. Prove that there exists some triangle with area less or equal than $\frac{1}{2}$ with its vertices among the points.

Proposition 2.2

Let $\mathbb{A}$ and $\mathbb{B}$ be rectangular prisms. Also assume that $\mathbb{B} \subseteq \mathbb{A}$ i.e. $\mathbb{B}$ completely lies inside the $\mathbb{A}$. Then the surface area of $\mathbb{A}$ is greater or equal than surface area of $\mathbb{B}$.

§2 Gaussian Measure

$\mathbb{R}^n$ with the Euclidean metric and with the n-dimensional Gaussian measure γ given by:

$$\gamma(A) = (2\pi)^{-n/2} \int_A e^{-||x||^2} dx$$

This is a probability measure on $\mathbb{R}^n$ corresponding to the n-dimensional normal distribution. Let $Z_1, Z_2, \ldots, Z_n$ be independent real random variables, each of them with the standard normal distribution $N(0,1)$, i.e., such that:

$$P[Z_i \leq z] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$$

Then the vector $(Z_1, Z_2, ..., Z_n) \in \mathbb{R}^n$ is distributed according to the measure γ.

This γ is spherically symmetric; the density function $e^{-||x||^2}$ depends only on the distance of x from the origin. The distance of a point chosen at random according to this distribution is sharply concentrated around $\sqrt{n}$, and in many respects, choosing a random point according to γ is similar to choosing a random point from the uniform distribution on the sphere $\sqrt{n}S^{n-1}$.

Theorem 2.4

Let a measurable set $A \subseteq \mathbb{R}^n$ *satisfy* $\gamma(A) \geq \frac{1}{2}$. *Then* $\gamma(A_t) \geq 1 - e^{-t^2/2}$.

Similar concentration inequalities also hold in many discrete metric spaces encountered in combinatorics. One of the simplest examples is the n-dimensional Hamming cube $C_n = \{0,1\}^n$. The points are n-component vectors of 0's and 1 's, and their Hamming distance is the number of positions where they differ. The "volume" of a set $A \subseteq \{0,1\}^n$ is defined as $P[A] = 2^{-n}|A|$.

An r-ball B is the set of all $0/1$ vectors that differ from a given vector in at most r coordinates, and so its volume is:

$$P[B] = 2^{-n}\left(\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}\right).$$

§2 The isoperimetric inequality for the Hamming cube

Theorem 2.5

If $A \subseteq C^n$ is any set with $P[A] > P[B]$, then $P[A_t] > P[B_t]$.

Theorem 2.6 (Measure concentration for the cube)

Let a measurable set $A \subseteq C^n$ *satisfy* $\gamma(A) \geq \frac{1}{2}$. *Then* $\gamma(A_t) \geq 1 - e^{-t^2/2n}$.

§3 Concentration of Lipschitz Functions

§3 Introduction

Main idea of this Section:

To show that any Lipschitz function on a high-dimensional sphere is tightly concentrated around its expectation.

A few notes:

1. We may consider any measurable real function $f : S^{n-1} \to \mathbb{R}$ as a random variable, with its expectation given by: $\mathbb{E}[f] = \int_{S^{n-1}} f(x) d\mathbb{P}(x)$.
2. A mapping f between matrix spaces is called C-Lipschitz for a $C > 0$ if $|f(y) - f(x)| < C|x - y| \ \forall x, y$.
3. Median of f is $med(f) = \sup\{t \in \mathbb{R} : \mathbb{P}[f \leq t] \leq \frac{1}{2}\}$.

§3 Lemma

Lemma 3.1

Let $f : \Omega \to \mathbb{R}$ *be a measurable function on a space* Ω *with a probability measure* $\mathbb{P}$ *then*

$$\mathbb{P}[f < med(f)] \leq \frac{1}{2} \text{ and } \mathbb{P}[f > med(f)] \leq \frac{1}{2}.$$

Proof.

The first inequality can be derived from countable additivity:

$$\mathbb{P}[f < med(f)] = \sum_{k=1}^{\infty} \mathbb{P}\left[med(f) - \frac{1}{k-1} < f \leq med(f) - \frac{1}{k}\right]$$

$$= \sup_{k \geq 1} \mathbb{P}\left[f \leq med(f) - \frac{1}{k}\right] \leq \frac{1}{2}.$$

The second inequality follows similarly. □

Theorem 3.2 (Lévy's Lemma)

Let $f : S^{n-1} \to \mathbb{R}$ be 1-Lipschitz then for all $t \in (0,1)$,

$$\mathbb{P}[f > med(f) + t] \leq 2e^{-t^2 n/2} \text{ and } \mathbb{P}[f < med(f) - t] \leq 2e^{-t^2 n/2}.$$

For example, on 99% of S^{n-1}, the function f attains values deviating from $med(f)$ by at most $3.5n^{-1/2}$.

Proof.

It is enough to show the first inequality. Let $A = \{x \in S^{n-1} : f(x) \leq med(f)\}$. By Lemma 3.1, $\mathbb{P}[A] > \frac{1}{2}$. Since f is 1-Lipschitz, we have $f(x) < med(f) + t$ for all $x \in A_t$. Therefore, by Theorem 1.1, we get

$$\mathbb{P}[f > med(f) + t] < 1 - P[At] \leq 2e^{-t^2 n/2}.$$

□

Proposition 3.1

Let $f : S^{n-1} \to \mathbb{R}$ be 1-Lipschitz then

$$|med(f) - \mathbb{E}[f]| \leq 12n^{-1/2}.$$

Proof.

$$|med(f) - \mathbb{E}(f)| \leq \mathbb{E}[|f - med(f)|] \leq \sum_{k=0}^{\infty} \frac{k+1}{\sqrt{n}} \mathbb{P}\left[|f - med(f)| \geq \frac{k}{\sqrt{n}}\right]$$

$$= n^{-1/2} \sum_{k=0}^{\infty} (k+1) 4 e^{-k^2/2} \leq 12n^{-1/2}.$$

□

§4 Almost Spherical Sections: The First Steps

§4 Introduction

Main idea of this Section:

Given a centrally symmetric convex body $K \subseteq \mathbb{R}^n$ and $\epsilon > 0$, we are interested in finding a k-dimensional (linear) subspace L, with k as large as possible, such that the "section" $K \cap L$ is $(1+\epsilon)$-almost spherical.

§4 Almost Spherical Sections: The First Steps

Definition 4.1

For a real number $t \geq 1$, we call a convex body K t-almost spherical if it contains a (Euclidean) ball B of some radius r and it is contained in the concentric ball of radius tr.

§4 Ellipsoid Lemma

Theorem 4.2

For any $(2k-1)$-dimensional ellipsoid E, there is a k-flat L passing through the center of E such that $E \cap L$ is a Euclidean ball.

Proof.

Let $E = \left\{ x \in \mathbb{R}^{2k-1} : \sum_{i=1}^{2k-1} \frac{x_i^2}{a_i^2} \leq 1 \right\}$ with $0 \leq a_1 \leq a_2 \leq \ldots \leq a_{2k-1}$. We define the k-dimensional linear subspace L by a system of $k-1$ linear equations. The i-th equation is :

$$x_i \sqrt{\frac{1}{a_i^2} - \frac{1}{a_k^2}} = x_{2k-1} \sqrt{\frac{1}{a_k^2} - \frac{1}{a_{2k-i}^2}}$$

$i = 1, 2, \ldots, k-1$.

§4 Proof

(Cont.)

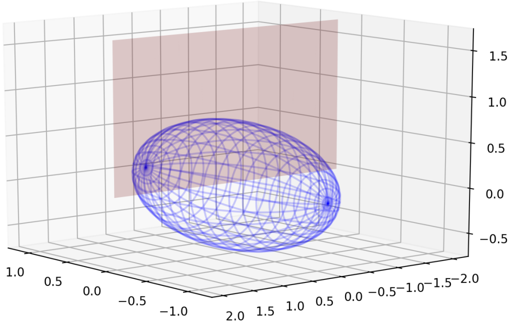
It is chosen so that

$$\frac{x_i^2}{a_i^2} + \frac{x_{2k-i}^2}{a_{2k-i}^2} = \frac{1}{a_k^2}\left(x_i^2 + x_{2k-i}^2\right)$$

for $x \in L$. It follows that for $x \in L$, we have $x \in E$ if and only if $||x|| < a_k$, and so $E \cap L$ is a ball of radius a_k . The reader is invited to find a geometric meaning of this proof and express it in the language of eigenvalues. $\square$

§4 Ellipsoid Lemma

Here is an example using the ellipsoid $\frac{x_1^2}{0.8^2} + \frac{x_2^2}{1.3^2} + \frac{x_1^2}{2^2} \leq 1$. The corresponding equation is $[0.98528370, 0, -0.5845648] \times [x_1, x_2, x_3] = [0, 0, 0]$, which yields the following shape:



§4 The Cube

The cube $[-1,1]^n$ is a good test case for finding almost-spherical sections; it seems hard to imagine how a cube could have very round slices. In some sense, this intuition is not totally wrong, since the almost-spherical sections of a cube can have only logarithmic dimension, as we verify next. (But the n-dimensional crosspolytope has $(1+\epsilon)$-spherical sections of dimension as high as $c(\epsilon)n$, and yet it does not look any rounder than the cube; so much for the intuition.) The intersection of the cube with a k-dimensional linear subspace of $\mathbb{R}^n$ is a k-dimensional convex polytope with at most 2k facets.

§4 Cube Lemma

Lemma 4.3

Let P be a k-dimensional 2-almost spherical convex polytope. Then P has at least $\frac{1}{2}e^{k/8}$ facets. Therefore, any 2-almost spherical section of the cube has dimension at most $O(\log n)$.

§4 Proof

Proof.

After a suitable affine transform, we may assume $\frac{1}{2}B_k \subseteq P \subseteq B_k$. Each point $x \in S^{k-1}$ is separated from P by one of the facet hyperplanes. For each facet F of P, the facet hyperplane h_F cuts off a cap C_F of S^{k-1} and these caps together cover all of S^{k-1}. The cap c_F is at distance at least $\frac{1}{2}$ from the hemisphere defined by the hyperplane h'_F parallel to h_F and passing through 0. By Theorem 14.1.1 (measure concentration), we have $P[C_F] < 2e^{-k/8}$. □

Lemma 5.1 (John's Lemma)

Let $K \subset \mathbb{R}^n$ be a bounded closed convex body with non-empty interior. Then there exists an ellipsoid E_{in} such that

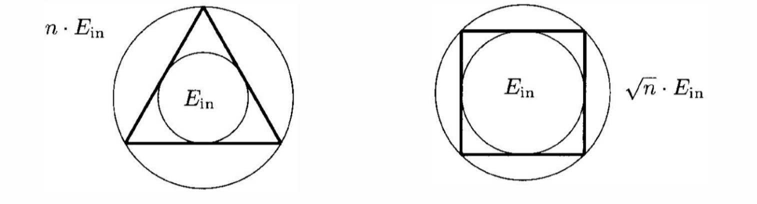
$$E_{in} \subseteq K \subseteq E_{out}$$

where E_{out} is E_{in} expanded from its center by the factor n. If K is symmetric about the origin, then we have the improved approximation

$$E_{in} \subseteq K \subseteq E_{out} = \sqrt{n} E_{in}.$$

Lemma 5.2 (John's Lemma Cont.)

Thus, K can be approximated from outside and from inside by similar ellipsoids with ratio $1 : n$, or $1 : \sqrt{n}$ for the centrally symmetric case. Both these ratios are the best possible in general, as is shown by K being the regular simplex in the general case and the cube in the centrally symmetric case.



§6 Dvoretzky's Theorem

Main idea of this Section:

Statement and informal proof of Dvoretzky's Theorem which suggests that any normed space of a sufficiently large dimension contains a large subspace on which the norm is very close to the Euclidean norm.

Theorem 6.1

For any natural number k and any real $\epsilon > 0$, there exists an integer $n = n(k, \epsilon)$ with the following property. For any n-dimensional centrally symmetric convex body $K \subseteq \mathbb{R}^n$, there exists a k-dimensional linear subspace $L \subseteq \mathbb{R}^n$ such that the section $K \cup L$ is $(1 + \epsilon)$-almost spherical. The best known estimates give $n(k, \epsilon) = e^{O(k/\epsilon^2)}$.

§6 Dvoretzky's Theorem: few notes

- Thus, no matter how "edgy" a high-dimensional K may be, there is always a slice of not too small dimension that is almost a Euclidean ball.
- We may also interpret the theorem as: Any normed space of a sufficiently large dimension contains a large subspace on which the norm is very close to the Euclidean norm.
- Note that the Euclidean norm is the only norm with this universal property, since all sections of the Euclidean ball are again Euclidean balls.
- The requirement for symmetricity may be replaced with having 0 in the interior of K.

In the proof we will consider the case $\epsilon = 1$.

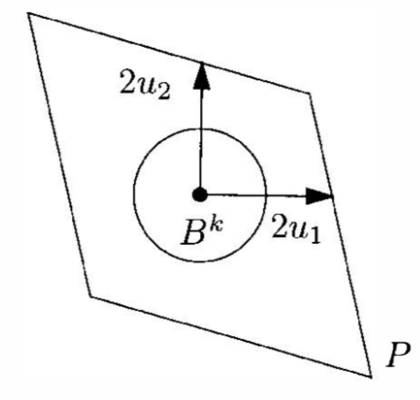
§6 Dvoretzky's Theorem: Dvoretzky-Rogers lemma

Lemma 6.2 (A version of the Dvoretzky-Rogers lemma)

Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body. Then there exist a linear subspace $Z \subset \mathbb{R}^n$ of dimension $k = \lfloor \frac{n}{\log_2 n} \rfloor$ an orthonormal basis $u_1, u_2, \ldots, u_k$ of Z and a nonsingular linear transform T of $\mathbb{R}^n$ such that if we let $K = T(K) \cap Z$, then $\|x\|_{\tilde{K}} \leq \|x\|$ for all $x \in Z$ and $\|u_i\|_{\tilde{K}} \geq \frac{1}{2}$ for all $i = 1, 2, \ldots, k$.

§6 Dvoretzky's Theorem: Dvoretzky-Rogers lemma

Geometrically, the lemma asserts that $\tilde{K}$ is sandwiched between the unit ball B^k and a parallelotope P as in the picture:



The lemma claims that the points $2u_i$ are outside of K or on its boundary, and P is obtained by separating these points from K by hyperplanes.

Proof.

By John's lemma, we may assume $B^n \subset K \subset tB^n$, where $t = \sqrt{n}$. Let $X_0 = \mathbb{R}^n$ and $K_0 = K$. Here is the main idea of the proof:

1. The current body K_i is enclosed between an inner ball and an outer ball. Either K_i approaches the inner ball sufficiently closely at "many" places,
2. and in this case we can construct the desired $u_1, \ldots, u_k$, or it stays away from the inner ball on a "large" subspace.
3. In the latter case, we can restrict to that subspace and inflate the inner ball. But since the outer ball remains the same, the inflation of the inner ball cannot continue indefinitely.

§6 Some other lemmas

Lemma 6.3

Let $\nu_1, \nu_2, \ldots, \nu_n$ be arbitrary vectors in a normed space with norm $\|\cdot\|$. Then

$$\sum_{\sigma_i \in \{-1,1\}^n} \left| \sum_{i=1}^{n} \sigma_i \nu_i \right| \geq 2^n \max_i |\nu_i|$$

Proof.

Without loss of generality let's assume $\max_i |v_i| = |v_1|$.Now we just group up our values to use triangle inequality. From scalable property we can just analyze for σ when $\sigma_1 = 1$. Then for any such σ, there is other unique such σ' where $\sigma + \sigma' = (2, 0, 0, \dots, 0)$. Then if we just apply multiple triangle inequalities we have the desired result. □

Lemma 6.4

For a suitable positive constant c and for all n we have

$$\mathbb{E}[f_C] = \frac{1}{2}\int_{S^{n-1}} \|x\|_\infty d\mathbb{P}(x) \geq c\sqrt{\frac{\log(n)}{n}},$$

where $\|x\|_\infty = \max_i |x_i|$ is the ℓ_∞ (or maximum) norm.

Proof.

Let $Z_1, Z_2, \ldots, Z_n$ be independent random variables, each of which have the standard normal distribution $N(0,1)$.The random vector $Z = (Z_1, Z_2, \ldots, Z_n)$ has a spherically symmetric gaussian distribution and the random variable $\frac{Z}{\|Z\|}$ is uniformly distributed in S^{n-1}. Hence we have

$$\mathbb{E}[f_C] = \frac{1}{2}\mathbb{E}\left[\frac{\|Z\|_\infty}{\|Z\|}\right]$$

We show first, that we have $\|Z\| < \sqrt{3}n$ with probability at least $\frac{2}{3}$ and second, that for a suitable constant $c_1 > 1$, $\|Z\|_\infty > c_1\sqrt{\log n}$ holds with probability at least $\frac{2}{3}$. It follows that both these events occur simultaneously with probability $\frac{1}{2}$ at least, it follows that $\mathbb{E}[f_C] \geq c\sqrt{\log n/n}$ as claimed. □

Proof.

- For $\|Z\|$, we may write $\mathbb{E}[\|Z\|] = n\mathbb{E}[Z_1^2] = n \cdot var(Z_1) = n$.
- $\mathbb{P}[\|Z\| \geq \sqrt{3n}] = \mathbb{P}\left[\|Z^2\| \geq 3\mathbb{E}[\|Z\|^2]\right] \leq \frac{1}{3}$.

Further, by the independence of the Z_i we have

$$\begin{aligned} \mathbb{P}[\|Z\|_\infty \leq z] &= \mathbb{P}[|Z_i| \leq z \; \forall \; i = 1, \ldots, n] \\ &= \mathbb{P}[|Z_1| \leq z]^n = \left(1 - \frac{2}{\sqrt{2\pi}} \int_z^\infty e^{-t^2/2} dt\right)^n . \end{aligned}$$

We can estimate $\int_z^\infty e^{-t^2/2} dt \geq \int_z^{z+1} e^{-t^2/2} dt \geq e^{-(z+1)^2/2}$. Thus setting $z = \sqrt{\log(n)} - 1$, we may obtain $\mathbb{P}\left[\|Z\|_\infty \leq z\right] \leq \left(1 - \frac{2}{\sqrt{2\pi}} n^{-1/2}\right)^n$ which is below $\frac{1}{3}$ for n sufficently large. □

Corollary 6.5 (Uniform distribution on the surface of unit sphere)

Let $Z_1, \dots, Z_n$ be iid with $Z_i \sim N(0,1)$, then $\frac{Z}{\sqrt{Z_1^2+\cdots+Z_n^2}}$ is distributed uniformly on the surface of the unit sphere.

Proof.

Using the previous lemma, we know that for an orthonormal matrix $\mathcal{O}$, Z is identically distributed with $\mathcal{O}Z$. Then $Y = \frac{Z}{\|Z\|}$ is identically distributed with $\frac{\mathcal{O}Z}{\|\mathcal{O}Z\|} = \frac{\mathcal{O}Z}{\|Z\|}$. It follows htat Y is invariant under rotations and belongs to the unit sphere which is only true of the uniform distribution. □

§6 Some other lemmas

Lemma 6.6 (Invariance of the Normal distribution under orthonormal transformations)

Let $X \sim N(\mu, \Sigma)$, $a \in \mathbb{R}^n$, and B be a $p \times p$ matrix, then $Y = a + BX \sim N(a + B\mu, B\Sigma B^T)$.

Proof.

We may write $X = \mu + AZ$ where A satisfies $\Sigma = AA^T$. Then $Y = a + BX = (a + B\mu) + (BA)Z$. □

References I

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Figure: Thank you!